Department of Mathematics:: I.I.T. Roorkee

Autumn Semester	**MAI-101: Mathematics - I**	**2024 – 2025**
Time: 180 Minutes	**End Term Exam**	Max. Marks. **100**

Note-A. Answer <u>ALL</u> questions.

Note-B. Write the Page No. details for each question in the front page of the main booklet of the answer script.

1. Let $A = \begin{bmatrix} \alpha & 1 & -1 \\ 2 & 5 & \beta \\ 1 & \gamma & 2 \end{bmatrix}$.

   Let one of the eigenvalues of A be 5 and its corresponding eigenvector be $(1, 2, 1)^T$.

   (a) Find the values of α, β, γ.

   (b) Find the remaining eigenvalues and corresponding eigenvectors of A.

   (c) Is A diagonalizable? If yes, find an invertible matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix. (3+5+2 Marks)

2. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 + 3xy^2}{3x^2 + y^2} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

   (a) Show that f is continuous at $(0, 0)$ using $\epsilon - \delta$ method.

   (b) Is f_x continuous at $(0, 0)$? Justify your answer.

   (c) Check the differentiability of f at $(0, 0)$? (4+4+4 Marks)

3-(a) Using Lagrange's multiplier method, find the shortest distance between the line $2x + y = 10$ and the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. (6 Marks)

(b) Let $F = F(x, y)$ and $x = e^u \cos t, y = e^u \sin t$. Find the expressions $\phi(u, t)$ and $\psi(u, t)$ such that

$$F_{uu} + F_{tt} = \phi(u,t)[F_{xx} + F_{yy}] + \psi(u,t)F_{xy}.$$

Given that $F_{xy} = F_{yx}$ and $F_{ut} = F_{tu}$. (6 Marks)

4-(a) Using the transformation $u = x + y$ and $v = x - y$, evaluate the integral

$$\iint_R (x-y)^{2024} \cos^2(x+y)\,dA$$

where R is the rhombus with its vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$. (7 Marks)

(b) Express

$$\int_0^1 y^{q-1}\left(\log_e\left(\frac{1}{y}\right)\right)^{p-1} dy, \qquad p,q>0$$

in terms of Beta or Gamma function. (4 Marks)

5-(a) Using spherical coordinates in triple integral, find the volume of the solid which is above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+(z-a)^2=a^2$. (6 Marks)

(b) Find the mass of the solid bounded by the surface

$$\left(\frac{x}{7}\right)^{2/3}+y^{2/3}+\left(\frac{z}{4}\right)^{2/3}=1,$$

where the density is being given by $\rho=10|xyz|$. (5 Marks)

6-(a) Find the volume of the region that lies inside of the surface $z=1-x^2-y^2$ and satisfying $z>1-y$. (6 Marks)

(b) Evaluate the surface integral

$$\int\int_S (x^2+y^2+z)dS$$

over the surface of the sphere $x^2+y^2+z^2=1$. (5 Marks)

7-(a) Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$. Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$ at the point $(1,2,-1)$. (4 Marks)

(b) Verify Green's theorem for evaluating $\oint_C y^2dx+xydy$, where C is the closed boundary of the semi-annular region between the semicircles $x^2+y^2=1$ and $x^2+y^2=4$ in the upper half plane. (7 Marks)

8-(a) Let $f(x,y,z)$ and $g(x,y,z)$ be two differentiable scalar fields such that

$$\nabla\cdot(f\nabla g)=2(\nabla f\cdot\nabla g)=12xyz.$$

Then, find the direction in which the scalar field $(f\nabla^2 g)$ has the maximum rate of change at the point $(1,1,1)$. (5 Marks)

(b) Find the value of real constant λ for which the vector field

$$\vec{F}=(y\sin z-\sin x)\hat{i}+(x\sin z+\lambda yz)\hat{j}+(xy\cos z+y^2)\hat{k}$$

is conservative. Hence, also find its scalar potential function. (6 Marks)

9-(a) Evaluate $\int\int_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F}=4xz\hat{i}+xyz^2\hat{j}+3z\hat{k}$ and S is the surface of the closed region above xy-plane bounded by the cone $z^2=x^2+y^2$ and the plane $z=2$.

(6 Marks)

(b) Using Stoke's theorem, evaluate the integral $\oint_C(ydx+zdy+xdz)$, where C is the boundary curve enclosing the surface $x^2+y^2+z^2=4$, $z\geq 0$. (5 Marks)

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