

Assignment-2: Matrix Algebra II

- (1) For each matrix, find all eigenvalues and the corresponding linearly independent eigenvectors;
- (a) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.
- (2) (a) Let λ be an eigenvalue of a nonsingular square matrix A of order n and \mathbf{x} be the corresponding eigenvector. Show that λ^{-1} is an eigenvalue of A^{-1} and identify the corresponding eigenvector. Also, identify the eigenvalue and eigenvector of $(A - kI)$, where I is the identity matrix and k is a scalar.
 (b) Let A be a square matrix of size n . Show that A and A^T have same eigenvalues. Are their eigenvectors also same?
 (c) If A and P are square matrices of order n , and P is nonsingular, then prove that A and $P^{-1}AP$ have the same eigenvalues.
- (3) Prove that
 (a) all eigenvalues of a Hermitian matrix are real.
 (b) eigenvalues of a skew Hermitian matrix are purely imaginary or zero.
 (c) eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.
 (d) eigenvalues of a unitary matrix have unit modulus.
 (e) any skew-symmetric matrix of odd order has zero determinant.
 (f) the eigenvalues of an idempotent matrix are either 0 or 1.
 (g) all eigenvalues of a nilpotent matrix are 0.
- (4) Let A and B be square matrices of order n . Show that if λ be an eigenvalue of AB then it will be an eigenvalue of BA . Hence, prove that $I - AB$ is invertible iff $I - BA$ is invertible.
- (5) Prove that every Hermitian matrix can be written as $A + iB$, where A is a real symmetric matrix, and B is a real skew-symmetric matrix.
- (6) Given that $A = \begin{bmatrix} 0 & 1 + 2i \\ -1 + 2i & 0 \end{bmatrix}$, show that $(I - A)(I + A)^{-1}$ is a unitary matrix.
- (7) (a) The eigenvalues of a 3×3 matrix A are 2, 2, 4 and the corresponding eigenvectors are $(-2, 1, 0)^T$, $(-1, 0, 1)^T$ and $(1, 0, 1)^T$. Find A .
 (b) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where A is
- (i) $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix}$.
- (8) Find e^{2A} and A^{50} when (a) $A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$
- (9) Using Cayley-Hamilton theorem, find the inverse of the following matrices:
- (a) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$
- (10) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$. Hence, find A^{-1} and A^4 .
- (11) Let $A = \begin{pmatrix} 4 & \alpha & -1 \\ 2 & 5 & \beta \\ 1 & 1 & \gamma \end{pmatrix}$. Given that the eigenvalues of the matrix A are 3, 3, δ (where $\delta \neq 3$) and A is diagonalizable, find the values of constants $\alpha, \beta, \gamma, \delta$.

- (12) Find an orthogonal or unitary matrix P such that P^*AP is diagonal where A is given by the following:

$$(a) \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \quad (b) \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

- (13) Let A be a 5×5 invertible matrix with row sums 1. That is $\sum_{j=1}^5 a_{ij} = 1$ for $1 \leq i \leq 5$.

Then, prove that the sum of all the entries of A^{-1} is 5.

- (14) Let A be a nilpotent matrix. Show that $I + A$ is invertible.

- (15) Suppose that $A^{15} = 0$. Show that there exists a unitary matrix U such that U^*AU is 5×5 upper triangular with diagonal entries 0.

ANSWERS

1. (a) $2, 2, 6, (-1, 0, 1)^T, (-1, 1, 0)^T, (1, 2, 1)^T$ (b) $1, 1, 1, (1, 0, 1)^T, (1, 0, 0)^T$
 (c) $-2, -2, 4, (0, 1, 1)^T, (1, 0, -1)^T, (1, 1, 2)^T$ (d) $1, 1, 3, (1, 0, 3)^T, (1, -3, 0)^T, (-1, 1, -1)^T$

7. (a) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & -11 & 1 \\ 1 & -1 & 1 \\ 1 & 14 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} \alpha & \beta - \alpha & (\alpha - 2\beta + \gamma)/2 \\ 0 & \beta & \gamma - \beta \\ 0 & 0 & \gamma \end{bmatrix}$ where $\alpha = e^{-8}, \beta = e^{-6}, \gamma = e^{-4}$ for e^{2A} and

$\alpha = 4^{50}, \beta = 3^{50}, \gamma = 2^{50}$ for A^{50}

(b) $\frac{1}{6} \begin{bmatrix} 6\alpha & 4(\beta - \alpha) & 3(\beta - \alpha) \\ 0 & 6\beta & 0 \\ 0 & 0 & 6\beta \end{bmatrix}$ where $\alpha = e^{-4}, \beta = e^8$ for e^{2A} and

$\alpha = 2^{50}, \beta = 4^{50}$ for A^{50}

9. (a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -0.20 & -0.90 & 0.50 \\ 0.50 & 1.25 & -0.75 \\ -0.30 & -0.60 & 0.50 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

10. $\frac{1}{11} \begin{bmatrix} 1 & -2 & 2 \\ 5 & 1 & -1 \\ -4 & 8 & 3 \end{bmatrix}, \begin{bmatrix} 25 & 8 & 8 \\ 12 & 25 & 4 \\ 16 & 32 & 33 \end{bmatrix}$

11. $\alpha = 1, \beta = -2, \gamma = 2, \delta = 5$.

12. (a) $\frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -5/\sqrt{105} & 4/\sqrt{21} \\ 1/\sqrt{5} & -8/\sqrt{105} & -2/\sqrt{21} \\ 2/\sqrt{5} & 4/\sqrt{105} & 1/\sqrt{21} \end{bmatrix}$ (c) $\begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{3} \\ (1+i)/\sqrt{6} & (1+i)/\sqrt{3} \end{bmatrix}$