Indian Institute of Technology Roorkee

MAI-101(Mathematics-I), Autumn Semester: 2024-25

Assignment-5: (Taylor's theorem, Maxima-Minima)

1. Let $f(x) = \sin x$. If $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$ then show that $\lim_{h\to 0} \theta = \frac{1}{\sqrt{3}}$.

2. Using Taylor's theorem, show that:
   (a) $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2}\, e^x, \;\; x > 0$
   (b) $x - \frac{x^3}{3!} < \sin x < x, \;\; x > 0$
   (c) $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, $x > 0$.

3. Obtain the Taylor's series expansion of the maximum order for the function $f(x,y) = x^2y + 3y - 2$ about the point $(1, -2)$.

4. Obtain Taylor's expansion of $\tan^{-1}\frac{y}{x}$ about $(1,1)$ up to second degree term. Hence compute $f(1.1, 0.9)$.

5. Find the linear approximation of the following functions at point P_0. Also, find the maximum absolute error in this approximation.
   (a) $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ at $P_0 = (-1, 1)$ and $R : |x+1| < 0.1, |y-1| < 0.1$.
   (b) $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at $P_0 = (3,2)$ and $R : |x-3| < 0.1, |y-2| < 0.1$.

6. Use Taylor's formula to find a quadratic approximation of $f(x,y) = \sin x \sin y$ at the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

7. Prove that the function $\left(\frac{1}{x}\right)^x$, $x > 0$ has a maximum at $x = e^{-1}$.

8. Examine the following functions for local extrema and saddle points:
   (a) $xy - x^2 - y^2 - 2x - 2y + 4$
   (b) $x^3 + y^3 - 3axy$
   (c) $x^2y^2 - 5x^2 - 8xy - 5y^2$
   (d) $2(x-y)^2 - x^4 - y^4$
   (e) $y \sin x$

9. Find the absolute maximum and minimum values of the following functions $f(x,y)$ in the closed region R:
   (a) $f(x,y) = 2 + 2x + 2y - x^2 - y^2$, R : triangular plate in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$.
   (b) $f(x,y) = 3x^2 + y^2 - x$, $R : 2x^2 + y^2 \le 1$.

10. Find the extrema of the following functions using the method of Lagrange multipliers:
    (a) $f(x,y) = 3x + 4y$ subject to the condition $x^2 + y^2 = 1$.
    (b) $f(x,y,z) = x^m y^n z^p$ subject to the condition $x + y + z = a$.
    (c) $f(x,y) = xy$ subject to the condition $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

11. Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral.

12. Find the shortest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

13. The plane $x+y+z = 1$ cuts the cylinder $x^2+y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

14. Find the quadratic approximation of $f(x, y) = \sqrt{1 + 4x^2 + y^2}$ at $(1, 2)$ and use it to compute the approximate value of $f(1.1, 2.05)$.

15. Find the maximum and minimum of $f(x, y) = (x+y)e^{-x^2-y^2}$ on the region defined by $x^2+y^2 \leq 1$.

Answers

3. $f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2)$.

4. $f(x, y) = \frac{\pi}{4} - \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{4}(y - 1)^2,\ 0.685$.

5. (a) $f(x, y) = -2 - 2(x + 1) - (y - 1),\ E(x, y) \leq 0.08$.
(b) $f(x, y) = 8 + 4(x - 3) - (y - 2),\ E(x, y) \leq 0.04$.

6. $f(x, y) = xy,\ E(x, y) \leq 0.0013$.

8. (a) Maximum value 8 at $(-2, -2)$.
(b) Maximum at (a,a) if $a < 0$, minimum at (a,a) if $a > 0$, and saddle point at (0,0).
(c) Maximum value 0 at $(0, 0)$, saddle point at $(\pm 1, \mp 1)$, $(\pm 3, \pm 3)$.
(d) Maximum value 8 at $(\pm\sqrt{2}, \mp\sqrt{2})$ and saddle point at $(0, 0)$.
(e) Saddle point at $(n\pi, 0)$ for $n \in I$.

9. (a) Absolute maximum value 4 at $(1, 1)$ and absolute minimum value -61 at (0,9) and (9,0).
(b) Absolute maximum value $\frac{3+\sqrt{2}}{2}$ at $(-1/\sqrt{2}, 0)$ and absolute minimum value -1/12 at (1/6,0).

10. (a) Maximum value 5 at $(3/5, 4/5)$ and minimum value -5 at $(-3/5, -4/5)$.
(b) $\frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$.
(c) Maximum value 2 at $(\pm 2, \pm 1)$ and minimum value -2 at $(\mp 2, \pm 1)$.

12. $\sqrt{6}$.

13. $(-1/\sqrt{2}, -1/\sqrt{2}, 1 + \sqrt{2})$ is farthest from the origin, $(1, 0, 0)$ and $(0, 1, 0)$ are closest to the origin.

14. 3.1691

15. Maximum value $\frac{1}{\sqrt{e}}$ at $(1/2, 1/2)$ and minimum value $-\frac{1}{\sqrt{e}}$ at $(-1/2, -1/2)$.