Department of Mathematics:: I.I.T. Roorkee

MAI-101(Mathematics-1) Autumn Semester: 2024-25

Assignment-7: Gamma and Beta Functions

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1. Evaluate (i) $\Gamma(7)$ (ii) $\Gamma\left(\frac{7}{2}\right)$.

2. Show that (i) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$

   (ii) $\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2m+1)}{2^{2m}\Gamma(m+1)}$, where where m is a non-negative integer.

   (iii) $2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2}) = \sqrt{\pi}\Gamma(2m)$, where m is a positive integer.

3. For $s > 0$, $p > 0$, show that

   (i) $\int_0^\infty x^{p-1}e^{-sx}dx = \Gamma(p)/s^p$ (ii) $\int_0^\infty e^{-s^2x^2}dx = \sqrt{\pi}/2s.$

4. Show that $\Gamma(p) = \int_0^1 (\ln(1/y))^{p-1}dy$; $p > 0$; and hence evaluate the integral $\int_0^1 (\ln(1/y))^{-1/2}dy.$

5. Show that for integer $m > -1, n > 0$
$$\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

6. Show that for $c > 1$, ,
$$\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\ln(c))^{c+1}}.$$

7. Show that for $r > -1$,
$$\int_0^\infty x^r e^{-s^2x^2}dx = \frac{1}{2s^{r+1}}\Gamma\left(\frac{r+1}{2}\right).$$

8. Prove the following: (i) $\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta.$

   (ii) $\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt.$

   (iii) $\beta(x,y) = \beta(x+1,y) + \beta(x,y+1).$

(iv) $\dfrac{1}{x+y}\beta(x,y) = \dfrac{1}{x}\beta(x+1,y) = \dfrac{1}{y}\beta(x,y+1)$.

(v) $\displaystyle\int_0^1 t^{m-1}(1-t^2)^{n-1}dt = \frac{1}{2}\beta\left(\frac{m}{2},n\right)$.

(vi) $\displaystyle\int_0^1 (1-t^6)^{-1/6}dt = \frac{\pi}{3}$.

9. Show that for any $m \in \mathbb{N}$,

$$\beta(m,m) = \frac{\sqrt{\pi}\,\Gamma(m)}{2^{2m-1}\Gamma(m+1/2)}.$$

10. Evaluate the following integrals in terms of Gamma and Beta functions:
(i) $\displaystyle\int_0^\infty e^{-x^4}dx$ (ii) $\displaystyle\int_0^\infty x^{-7/4}e^{-\sqrt{x}}dx$ (iii) $\displaystyle\int_0^a x^9(a^6-x^6)^{1/3}dx$, $a > 0$.

11. Evaluate $\displaystyle\int_0^\infty xe^{-x}\cos\alpha x\,dx$, $\alpha > 0$, using Gamma integral, and hence find the value of $\displaystyle\int_0^\infty x^2e^{-x}\sin\alpha x\,dx$.

12. Evaluate $\displaystyle\int_0^\infty xe^{-x}\sin\alpha x\,dx$, $\alpha > 0$, using Gamma integral, and hence find the value of $\displaystyle\int_0^\infty x^2e^{-x}\cos\alpha x\,dx$.

13. Prove that

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\prod_{k=1}^{n}\frac{2k-1}{2} \text{ for } n \in \mathbb{N}.$$

14. Show that (Dirichlet's Integral)

$$\iiint_V x^2y\,dV = \frac{\pi}{8},$$

where V is the region bounded by $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$ and the coordinate planes.

Answers.

(1) (i) 720 (ii) $\dfrac{15}{8}\sqrt{\pi}$ (4) $\sqrt{\pi}$

(10) (i) $\Gamma\left(\dfrac{5}{4}\right)$ (ii) $\dfrac{8}{3}\sqrt{\pi}$ (iii) $\dfrac{a^{12}}{6}\beta\left(\dfrac{5}{3},\dfrac{4}{3}\right)$

(11) $\dfrac{(1-\alpha^2)}{(1+\alpha^2)^2}, \dfrac{2\alpha(3-\alpha^2)}{(1+\alpha^2)^3}$ (12) $\dfrac{2\alpha}{(1+\alpha^2)^2}, \dfrac{2(1-3\alpha^2)}{(1+\alpha^2)^3}$