

**Indian Institute of Technology Roorkee**  
**MAI-101(Mathematics-1)**  
**Autumn Semester: 2024-25**  
**Assignment-10: Vector Calculus II**  
**(Line and surface integrals, Green's, Gauss and Stokes's theorem and their applications)**

1. Evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = x^2 \hat{i} - xz \hat{j} + y^2 \hat{k}$  along the path C joining the points  $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1) \rightarrow (0,0,1)$  via straight lines.
2. Show that  $\vec{F} = (y^2 \cos x + z^3) \hat{i} + (2y \sin x - 4) \hat{j} + (3xz^2 + 2) \hat{k}$  is a conservative vector field and find a function  $\phi$  such that  $\vec{F} = \nabla \phi$ . Also, find the work done by a moving particle from  $(0, 1, -1)$  to  $(\pi/2, -1, 2)$ .
3. If  $\vec{F} = \left( \frac{x \hat{j}}{x^2 + y^2} - \frac{y \hat{i}}{x^2 + y^2} \right)$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  for the various curves C from  $(0, 1)$  to  $(1, 0)$  along
  - (i) the arc of  $x^2 + y^2 = 1$  lying in the second, third and fourth quadrant.
  - (ii)  $x + y = 1$ .
  - (iii) the arc of  $x^2 + y^2 = 1$  lying in the first quadrant.

Is the vector field  $\vec{F}$  conservative? If so, find  $\phi$  such that  $\nabla \phi = \vec{F}$ . Why is the line integral not path independent?
4. Evaluate the surface integral  $\iint_S \vec{F} \cdot \hat{n} \, dS$ , if
  - (i)  $\vec{F} = yz \hat{i} + zx \hat{j} + xy \hat{k}$  and S is the surface of  $x^2 + y^2 + z^2 = 1$  in the first octant.
  - (ii)  $\vec{F} = z \hat{i} + x \hat{j} - 3y^2 z \hat{k}$  and S is the surface of  $x^2 + y^2 = 16$  in the first octant between  $z = 0$  and  $z = 5$ .
  - (iii)  $\vec{F} = \frac{\vec{r}}{r^3}$  and S is the surface of  $x^2 + y^2 + z^2 = a^2$ .
5. If  $\vec{F} = 4xz \hat{i} + xy z^2 \hat{j} + 3z \hat{k}$ , evaluate the volume integral  $\iiint_V \nabla \cdot \vec{F} \, dV$  over the entire surface of the region above the xy- plane bounded by the cone  $z^2 = x^2 + y^2$  and the plane  $z = 4$ .
6. Evaluate  $\iiint_V \phi \, dV$ , where  $\phi = 45x^2y$  and V is the closed region bounded by the planes  $4x + 2y + z = 8$ ,  $x = 0$ ,  $y = 0$  and  $z = 0$ .

7. Evaluate  $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$ , where  $\vec{F} = y^2 \hat{i} + y \hat{j} - xz \hat{k}$  and  $S$  is the surface of the sphere  $x^2 + y^2 + z^2 = a^2$  above  $xy$ -plane.
8. Verify Green's theorem for
- $\oint_C [(xy^2 - 2xy)dx + (x^2y + 3)dy]$  around the boundary curve  $C$  of the region enclosed by  $y^2 = 8x$  and  $x = 2$ .
  - $\oint_C [(xy + y^2)dx + x^2dy]$ ,  $C$  bounds the region enclosed by  $y = x$  and  $y = x^2$ .
  - $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$  and  $C$  bounds the region enclosed by  $x = 0$ ,  $y = 0$  and  $x + y = 1$ .
9. By converting into the line integral, evaluate  $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$ ,  
where  $\vec{F} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$  and  
 $S$  is the surface of the cone  $z = 2 - \sqrt{x^2 + y^2}$  above  $xy$ -plane.
10. Verify Stoke's theorem for  $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$  taken around the rectangle bounded by the lines  $x = \pm a$ ,  $y = 0$  and  $y = b$ .
11. Verify Gauss's divergence theorem for
- $\vec{F} = (2x - z)\hat{i} - x^2y\hat{j} + 4xz^2\hat{k}$  taken over the region bounded by the planes  $x = 0$ ,  $x = 1$ ,  $y = 0$ ,  $y = 1$ ,  $z = 0$  and  $z = 1$ .
  - $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$  taken over the region in the first octant bounded by  $y^2 + z^2 = 9$  and  $x = 2$ .
12. Evaluate  $\iint_S \vec{F} \cdot \vec{n} \, ds$  where  $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$  and  $S$  is a rectangular parallelepiped  $0 \leq x \leq a$ ,  $0 \leq y \leq b$  and  $0 \leq z \leq c$ .

### Answers

- (1)  $3/2$  (2).  $\phi = y^2 \sin x + xz^3 - 4y + 2z$ ,  $4\pi + 15$
- (3).  $\frac{3\pi}{2}$ ,  $\frac{\pi}{2}$ ,  $-\frac{\pi}{2}$ ,  $\phi = \tan^{-1}(y/x)$  (4). (i)  $3/8$  (ii)  $90$  (iii)  $4\pi$
- (5).  $320\pi$  (6).  $128$  (7).  $0$  (9).  $12\pi$
- (12)  $abc(a + b + c)$