Indian Institute of Technology Roorkee
MAI-101(Mathematics-1), Autumn Semester: 2024-25
Assignment-1: Matrix Algebra I

(1) Reduce each of the following matrices into row echelon form and then find their ranks:

(a) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & 6 \\ -1 & 3 & 2 \\ 1 & 4 & 6 \\ 2 & 8 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$

(2) Examine the following set of vectors over $\mathbb{R}$ for linear dependence:

(a) $\{(1,0,0),\ (0,1,0),\ (1,1,1),\ (-1,1,-1)\}$ (b) $\{(1,-1,1),\ (2,1,1),\ (8,1,5)\}$

(c) $\{(1,-1,2,4),\ (2,-1,5,7),\ (-1,3,1,-2)\}$ (d) $\{(1,2,1),\ (2,1,0),\ (1,-1,2)\}$

(3) (a) Find the conditions on α and β for which the matrix

$\begin{pmatrix} \alpha & 1 & 2 \\ 0 & 2 & \beta \\ 1 & 3 & 6 \end{pmatrix}$ has (i) rank $= 1$ (ii) rank $= 2$ (iii) rank $= 3$.

(b) For what values of α and β is the following system consistent?

$2x + 4y + (\alpha+3)z = 2,\ \ x + 3y + z = 2,\ \ (\alpha-2)x + 2y + 3z = \beta.$

(4) Solve the following system of linear equations by Gauss elimination method:

(a) $x + 4y - z = 4,\ x + y - 6z = -4,\ 3x - y - z = 1$

(b) $x + y + z = -3,\ 3x + y - 2z = -2,\ 2x + 4y + 7z = 7$

(c) $x + 2y + z = 2,\ 3x + y - 2z = 1,\ 2x + 4y + 2z = 4$

(d) $2\sin x - \cos y + 3\tan z = 3,\ 4\sin x + 2\cos y - 2\tan z = 10,\ 6\sin x - 3\cos y + \tan z = 9$

(5) Consider the following systems of linear equations:

(a) $2x + 3y + 5z = 9,\ 2x + 3y + rz = s,\ 7x + 3y - 2z = 8$

(b) $x + y - z = 1,\ 2x + 3y + \lambda z = 3,\ x + \lambda y + 3z = 2$

(c) $\lambda x + y + z = p,\ x + \lambda y + z = q,\ x + y + \lambda z = r$

Find the values of unknown constant(s) such that each of the above systems has (i) no solution (ii) a unique solution (iii) infinitely many solutions.

(6) Let P_2 be the set of all polynomials of degree 2 or less. Use Gauss elimination method to find polynomial(s) $f \in P_2$ such that $f(0) = 1, f(1) = 2$ and $f(-1) = 6$.

(7) Find the values of k for which the following system of equations has (i) trivial solution (ii) non-trivial solution.

(a) $\begin{aligned} (3k-8)x + 3y + 3z &= 0 \\ 3x + (3k-8)y + 3z &= 0 \\ 3x + 3y + (3k-8)z &= 0 \end{aligned}$ (b) $\begin{aligned} (k-1)x + (3k+1)y + 2kz &= 0 \\ (k-1)x + (4k-2)y + (k+3)z &= 0 \\ 2x + (3k+1)y + 3(k-1)z &= 0 \end{aligned}$

(8) By employing elementary row operations, find the inverse of the following matrices:

(a) $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 1 & 3 \end{pmatrix}$

(9) Suppose $X, Y \in \mathbb{R}^n$, $n > 1$ are any two column matrices. Prove or disprove that the matrix $A = XY^T$ is invertible.

(10) Find the value of θ for which the following system of equations has non-trivial solution?

$2(\sin\theta)x + y - 2z = 0,\ 3x + 2(\cos 2\theta)y + 3z = 0,\ 5x + 3y - z = 0.$

(11) (a) Let A be an $n \times n$ matrix. If A is not invertible, then prove that there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

(b) Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & 1 & 0 & 1 \\ 0 & 5 & -2 & 7 \\ -1 & 3 & -1 & 4 \end{bmatrix}$. Find a 4×4 matrix $B \neq 0$ such that $AB = 0$.

(12) Consider a 4×5 matrix $A = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 3 & 21 & 0 & 9 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$

(a) Find the row-reduced echelon form of A.

(b) Find an invertible matrix P such that $PA = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix}$

(c) Find the locus of the point $(x, y, z) \in \mathbb{R}^3$ such that for each column vector $Y = (x, y, z, 5)^T$, the equation $AX = Y$ has a solution.

(d) If $X = (x_1, x_2, x_3, x_4, x_5)^T$, then find the conditions on x_1, x_2, x_3, x_4, x_5 such that $AX = 0$.

ANSWERS

(1) (a) 3 (b) 3 (c) 2 (d) 3

(2) (a) LD (b) LD (c) LI (d) LI

(3) (a) (i) Not possible (ii) $\alpha = \frac{1}{3}$ or $\beta = 4$ (iii) $\alpha \neq \frac{1}{3},\ \beta \neq 4$
(b) $\alpha = 3$ and $\beta = 1$; or $\alpha = -2$ and $\beta = 6$; or $\alpha \neq 3, -2$.

(4) (a) (1,1,1) (b) No solution (c) Infinite solutions (d) No solution

(5) (a) (i) $r = 5, s \neq 9$ (ii) $r \neq 5, s \in \mathbb{R}$ (iii) $r = 5, s = 9$.
(b) (i) $\lambda = -3$ (ii) $\lambda \neq -3, 2$ (iii) $\lambda = 2$
(c) (i) $\lambda = 1$ and $p + q - 2r \neq 0$ OR $\lambda = 1$ and $q \neq r$ OR $\lambda = 1$ and $r \neq p$ OR $\lambda = 1$ and $p \neq q$ OR $\lambda = -2$ and $p + q + r \neq 0$ and $q \neq r$
(ii) $\lambda \neq 1, -2$
(iii) $\lambda = 1$ and $p = q = r$ OR $\lambda = -2$ and $p + q + r = 0$

(6) $f(x) = 3x^2 - 2x + 1$

(7) (a) (i) $k \neq \frac{2}{3}, \frac{11}{3}$ (ii) $k = \frac{2}{3}$ or $\frac{11}{3}$ (b) (i) $k \neq 0,\ 3$ (ii) $k = 0$ or 3

(8) (a) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} -16 & 4 & -4 & 12 \\ 5 & -1 & -1 & 0 \\ 9 & -1 & 3 & -8 \\ 6 & -2 & 2 & -4 \end{bmatrix}$

(10) $n\pi + (-1)^n \frac{\pi}{6}$ or $n\pi + (-1)^n \sin^{-1}(9 - \sqrt{161})/4,\ n = 0, 1, 2, \ldots$

(11) (b) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -7 & 2 & -5 \\ 5 & 0 & 5 & 5 \\ 0 & 5 & 0 & 5 \end{bmatrix}$ (This is just one solution. The matrix B is not unique).

(12) (a) $\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \end{bmatrix}$

(c) $x + y + z = 5$

(d) $x_1 + 7x_2 + 3x_4 = 0, \quad x_3 + 5x_4 = 0, \quad x_5 = 0.$