Indian Institute of Technology Roorkee
MAI-101(Mathematics-1), Autumn Semester: 2024-25

<u>Assignment-3: Differential Calculus</u>

(1) Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. Show that each of the following functions are continuous on $\mathbb{R}^2$:
(a) $f - g$
(b) fg
(c) $\max\{f, g\}$
(d) $\min\{f, g, h\}$

(2) Find the following limits, if they exist
(a) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$ (c) $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$
(d) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (e) $\lim_{(x,y)\to(0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$ (f) $\lim_{(x,y)\to(1,1)} f(x,y)$ where
$f(x,y) = \begin{cases} 1, & \text{if } x+y \geq 2 \\ -1, & \text{if } x+y < 2 \end{cases}$ (g) $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{1}{xy}$

(3) (a) Consider the function $f(x,y) = \frac{x+y}{x-y}$ for $(x,y) \in \mathbb{R}^2$ with $x - y \neq 0$. Show that $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)] = 1$, but $\lim_{y\to 0}[\lim_{x\to 0} f(x,y)] = -1$. What can you say about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$?

(b) Let $f(x,y) = 0$ if $y = 0$, and $f(x,y) = x\sin\left(\frac{1}{y}\right)$, if $y \neq 0$. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ and iterated limits $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)]$ and $\lim_{y\to 0}[\lim_{x\to 0} f(x,y)]$ if they exist.

(c) Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ if $x^2y^2 + (x-y)^2 \neq 0$. Show that $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)]$ and $\lim_{y\to 0}[\lim_{x\to 0} f(x,y)] = 0$. But $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(4) Let $f(x,y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$. Show that for any point (a,b), $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

(5) Examine the continuity of the function $f(x,y)$ at $(0,0)$ in each of the following cases. Also check the existence of $f_x(0,0)$ and $f_y(0,0)$.
(a) $f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & x+2y \neq 0 \\ \frac{1}{2}, & x+2y = 0 \end{cases}$ (b) $f(x,y) = \sqrt{|xy|}$
(c) $f(x,y) = \begin{cases} xy\log(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$
(d) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y. \end{cases}$

(6) For the function $f(x,y) = \begin{cases} \frac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.
Compute $f_x(0,y)$, $f_y(x,0)$, $f_x(0,0)$ and $f_y(0,0)$, if they exist.

(7) Show that the function $f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2}, & \text{when } xy \neq 0 \\ x^3 \sin\frac{1}{x^2}, & \text{when } x \neq 0 \text{ and } y = 0 \\ y^3 \sin\frac{1}{y^2}, & \text{when } x = 0 \text{ and } y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$ is differentiable at $(0,0)$, whereas none of f_x, f_y is continuous at $(0,0)$.

(8) Determine the values of p for which $f(x,y) = |xy|^p$, $xy \neq 0$, and $f(x,y) = 0$, $xy = 0$, is continuous and differential at $(0,0)$.

(9) Show that the function $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$ and that f_x, f_y are not continuous at $(0,0)$.

(10) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable everywhere except at $(0,0)$. Suppose that all partial derivatives of f exist at origin.
   (a) Can f be extended to a continuous map from $\mathbb{R}^2 \to \mathbb{R}$?
   (b) Suppose f is continuous at origin. Is f differentiable from $\mathbb{R}^2 \to \mathbb{R}$?

(11) Check differentiablility of the following functions:
   (a) $f(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.
   (b) $f(x,y) = 1$ if $0 \leq y \leq x^2$ and $f(x,y) = 0$ otherwise.
   (c) $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ for $(x,y) \neq 0$, and $f(0,0) = 0$.
   (d) $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at $(0,0)$.
   (e) $f(x,y) = \dfrac{x^2 y^2}{x^4+y^2}$ for $(x,y) \neq 0$, and $f(0,0) = 0$.

Answers:

2. (a) 0 (b) does not exist (c) does not exist (d) 0 (e) 0 (f) does not exist (g) 0
3. (a) does not exist (b) 0, does not exist, 0
5. (a) continuous, 0,0 (b) continuous, 0,0 (c) continuous, 0,0
   (d) discontinuous, 0,0
6. 0, 1, 0, -1.
8. For continuity $p > 0$ and for differentiability $p > 1/2$.
10. Not in general for both cases.
11. No, No, No, Yes, Yes.