**Indian Institute of Technology Roorkee**

**MAI-101(Mathematics-1), Autumn Semester: 2023-24**

**Assignment-4: (Euler's theorem, Chain Rule, Jacobian)**

1. If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e x - \log_e y}{x^2+y^2}$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$.

2. Let $u = \cos^{-1}\left(\frac{x^2+y^2}{\sqrt{x}-\sqrt{y}}\right)$. Then by using Euler's theorem, prove that

   (i) $2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} + 3\cot u = 0$. (ii) $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{3}{2}\cot u(1 - \frac{3}{2}\csc^2 u)$.

3. Let $z = x^2y^4 sin^{-1}(\frac{x}{y})$, where $0 < x < y$. Determine $m$, if $x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} = mz$.

4. If $z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{y}{x}\right)$, then show that $x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$.

5. Let $u(x,y) = \ln(\frac{x^3+y^3}{x+y}) + \tan^{-1}(\frac{x^3+y^3}{x-y})$. Then, using Euler's theorem, evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

6. Use the chain rule to compute $\frac{du}{dt}$ if

   (i) $u = \sin(x^2+y^2),\ x = t^2+3,\ y = t^3$.

   (ii) $u = \tan^{-1}(\frac{y}{x}),\ x = e^t - e^{-t},\ y = e^t + e^{-t}$.

   (iii) $u = x^2+y^2+z^2$ and $x = e^{2t},\ y = e^{2t}\cos 3t,\ z = e^{2t}\sin 3t$.

7. Use the chain rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = \cos(x^3+y^3),\ x = st,\ y = t^2+s^2$.

8. Find the first order partial derivatives of $z$ with respect to $x$ and $y$ if $xy + yz + xz = 1$.

9. If $z = e^x \sin y + e^y \cos x$, where $x$ and $y$ are implicit functions of $t$ defined by $x^3+x+e^t+t^2+t-1 = 0$ and $yt^3 + y^3t + t + y = 0$, then find $\frac{dz}{dt}$ at $t = 0$.

10. Find the values of $n$ so that the function $v = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}(r^2\frac{\partial v}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial v}{\partial\theta}) = 0.$$

11. Find the positive value of $n$ so that the function $f(r,\theta) = r^n e^\theta \cos 2\theta$ satisfies the relation

$$\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) - 2\frac{\partial^2 f}{\partial r \partial\theta} + n\frac{\partial}{\partial\theta}(\frac{1}{r}\frac{\partial f}{\partial\theta}) = 0.$$

12. If $v = v(r)$, where $r^2 = \sum_{i=1}^n x_i^2$, show that

$$\sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r}.$$

13. Let $w = f(u,v)$ satisfy the Laplace equation $w_{uu} + w_{vv} = 0$. If $u = \frac{x^2-y^2}{2}$ and $v = xy$, then show that $w$ also satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

14. Let $f(x,y)$ be a function having continuous second order partial derivatives. If $x = \alpha\cosh u \cos v$ and $y = \alpha\sinh u \sin v$, where $\alpha$ is a constant, then show that

$$f_{uu} + f_{vv} = \frac{\alpha^2}{2}(\cosh 2u - \cos 2v)(f_{xx} + f_{yy}).$$

15. If $x = r\cos\theta, y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\left(\frac{\partial^2 r}{\partial x^2}\right) \cdot \left(\frac{\partial^2 r}{\partial y^2}\right) = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$

(iii) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

Further, if $V$ is a function of $x$ and $y$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$.

16. If $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi,\ z = r\cos\theta$ and $U = U(x, y, z)$, then prove that

$$\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 = \left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\right)^2.$$

17. If $u = \log_e\left(x^3 + y^3 + z^3 - 3xyz\right)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

18. Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$, where $x = \rho\sin\theta\cos\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\theta$.

19. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ where $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v,\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$.

20. Check whether the functions are functionally dependent or not? If yes, then find a relation between them.

(i) $f(x, y) = \log x - \log y,\ g(x, y) = \frac{x^2+3y^2}{2xy}$. (ii) $f(x, y) = \frac{y}{x},\ \ g(x) = \frac{x-y}{x+y}$.

21. Show that the following functions satisfy the necessary condition for functional dependence

$$u = x + y + z, \quad v = x^2 + y^2 + z^2, \quad w = x^3 + y^3 + z^3 - 3xyz.$$

Also find a relation among $u, v, w$.

22. Let $f(x, y) = e^x\cos y$ and $g(x, y) = x + \ln(\cos y)$ for $x \in \mathbb{R}$ and $0 < y < \frac{\pi}{2}$. Then, find the Jacobian $\frac{\partial(f,g)}{\partial(x,y)}$, and use it to check whether the functions are functionally dependent or not. If yes, then find a relation between them.

23. If the roots of equation $(t - x)^3 + (t - y)^3 + (t - z)^3 = 0$ in $t$ are $u, v, w$, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}.$$

## Answers

**3.** 30 **6.** (i) $4xt\cos(x^2 + y^2) + 6yt^2\cos(x^2 + y^2)$, (ii) $-2/(e^{2t} + e^{-2t})$, (iii) $8e^{4t}$.

**8.** $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y},\ \frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$. **9.** $-2$ **10.** $n = 2, -3$ **11.** $n = 5$ **18.** $\rho^2\sin\theta$ **19.** $\frac{4}{\sqrt{3}}$

**20.** (i) dependent. $f(x, y) = \log(g(x, y)) + \sqrt{(g(x, y))^2 - 3})$
(ii) dependent. $f(x, y) = \frac{1-g(x,y)}{1+g(x,y)}$

**21.** $w = \frac{u(3v-u^2)}{2}$. **22.** 0, dependent, $\ln(f(x, y)) - g(x, y) = 0$