Indian Institute of Technology Roorkee
MAI-101(Mathematics I)
Autumn Semester 2024–25
Assignment 6: (Multiple Integrals)

1. Sketch the region R in the xy-plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

2. Consider
$$f(x,y) = \frac{x-y}{(x+y)^3}; \quad 0 \le x \le 1,\ 0 \le y \le 1.$$
(a) Evaluate the repeated integrals $\int_0^1 dx \int_0^1 f(x,y)dy$ and $\int_0^1 dy \int_0^1 f(x,y)dx$.
(b) Comment on the existence of the double integral.

3. Evaluate the following integrals by interchanging the order of integration:
(a) $\int_0^1 \int_{4y}^4 e^{x^2} dxdy$. (b) $\int_0^1 \int_{y^2}^1 (ye^{x^2})dxdy$. (c) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx$.
(d) $\int_0^8 \int_{y^{\frac{1}{3}}}^2 \sqrt{(x^4+1)}dxdy$. (e) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy$. (f) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}} dydx$.

4. Evaluate:
(a) $\int\int_D (4x+2)dA$, where D is a region enclosed by the curves $y = x^2$ and $y = 2x$.
(b) $\int\int_R [x+y]dA$, over the rectangle formed by the coordinate axes and the lines $x = 1$, $y = 2$.

5. Evaluate the following double integrals:
(a) $\int\int_R (x^2+y^2)dA$, where R is the region of the plane given by $x^2+y^2 \le a^2$.
(b) $\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}dxdy$.

6. Show the followings by changing the order of integration:
(a) $\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r,\theta)drd\theta = \int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r,\theta)d\theta dr$.
(b) $\int_0^{\pi/3} \int_{a\sec^2(\theta/2)}^{(8a/3)\cos\theta} f(r,\theta)drd\theta = \left[\int_a^{4a/3} \int_0^{2\cos^{-1}(\sqrt{a/r})} + \int_{4a/3}^{8a/3} \int_0^{\cos^{-1}(3r/8a)}\right] f(r,\theta)d\theta dr$.

7. Prove that
(a) $\int_0^a \int_0^x \frac{f'(y)dydx}{\sqrt{(a-x)(x-y)}} = \pi(f(a) - f(0))$.
(b) $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{bdydx}{(x^2+y^2+b^2)^{3/2}(x^2+y^2+a^2)^{1/2}} = \frac{2\pi}{a+b}$. (By changing into polar coordinates.)

8. Evaluate the following triple integrals:
(a) $\int\int\int_E 2xdV$, where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.
(b) $\int\int\int_E \sqrt{3x^2+3z^2}dV$, where E is the solid bounded by $y = 2x^2+2z^2$ and the plane $y = 8$.

(c) $\int\int\int_E xyz dV$, where E is the solid bounded by the sphere of radius 2 in the first octant.

(d) $\int\int\int_E dV$, where E is the solid bounded by the cylinder $x^2+y^2=9$ and the planes $z=1$ and $x+z=5$.

9. Find the volume of the region bounded above by the paraboloid $z=5-x^2-y^2$ and below by the paraboloid $z=4x^2+4y^2$.

10. Evaluate the following integrals by changing the variables into cylindrical coordinates:

(a) $\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{16-x^2-y^2}\sqrt{x^2+y^2}dzdydx$.

(b) $\int\int\int_E\sqrt{(x^2+y^2)}dV$, where E is the region lying above the xy-plane and below the cone $z=4-\sqrt{x^2+y^2}$.

(c) $\int\int\int_E dV$, where E is the region bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the plane $z=b$, where $a>b>0$.

11. By using spherical coordinates evaluate the following triple integrals:

(a) $\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 dzdydx$.

(b) $\int\int\int_E(x^2+y^2+z^2)^{\frac{1}{2}}dV$, where E is the region bounded by the plane $z=3$ and the cone $z=\sqrt{x^2+y^2}$.

(c) $\int\int\int_E(x^2+y^2+z^2)^{-\frac{3}{2}}dV$, where E is the region bounded by the spheres of radius 2 and 3.

12. Evaluate $\int\int_R(\frac{x-y}{x+y+2})^2dxdy$, where R is the region bounded by the lines $x+y=\pm1,\ x-y=\pm1$. (Use the transformation $u=x+y,\ v=x-y$ and integrate over an appropriate region in $uv-$plane.)

13. Evaluate $\int\int_R(3x^2+14xy+8y^2)dxdy$, where R is the region in the first quadrant bounded by the lines $y=-\frac{3}{2}x+1$, $y=-\frac{3}{2}x+3$, $y=-\frac{1}{4}x$ and $y=-\frac{1}{4}x+1$, using the transformation $u=3x+2y$ and $v=x+4y$.

14. Evaluate $\int\int_R e^{x^2-y^2}dA$, where R is the region in the first quadrant bounded by $x^2-y^2=1,\ x^2-y^2=4, y=0$ and $y=(3/5)x$, by using the transformation $u=x^2-y^2$ and $v=x+y$.

15. Evaluate $\int_0^3\int_0^4\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}(\frac{2x-y}{2}+\frac{z}{3})dxdydz$ by applying the transformation $u=\frac{2x-y}{2},\ v=\frac{y}{2},\ w=\frac{z}{3}$, and integrating over an appropriate region in $uvw-$plane.

Answers:

1. $\frac{2}{3}$.

2. (a) $\frac{1}{2}$, $-\frac{1}{2}$. (b) Does not exist.

3. (a) $\frac{1}{8}(e^{16}-1)$. (b) $\frac{1}{4}(e-1)$. (c) 1. (d) $\frac{1}{6}(17^{\frac{3}{2}}-1)$. (e) $\frac{241}{60}$. (f) $\frac{\pi a^2}{6}$.

4. (a) 8. (b) 3 .

5. (a) $\frac{\pi a^4}{2}$. (b) $\frac{4\pi}{9}$.

8. (a) 9. (b) $\frac{256\sqrt{3}\ \pi}{15}$. (c) $\frac{4}{3}$. (d) 36π.

9. $\frac{5\pi}{2}$.

10. (a) $\frac{1024(\pi)}{15}$. (b) $\frac{64}{3}(2\pi)$. (c) $\frac{\pi}{3}(a^3 - 3a^2b + b^3)$.

11. (a) $\frac{\pi}{3}$, (b) $\frac{27\pi}{2}(2\sqrt{2} - 1)$, (c) $4\pi \log(\frac{3}{2})$.

12. $\frac{2}{9}$. **13.** $\frac{64}{5}$. **14.** $\frac{\log 2}{2}(e^4 - e)$. **15.** 12.