ETE- MAI:101

Solution with step Marking

Q.1

(a)

$$\begin{bmatrix} \alpha & 1 & -1 \\ 2 & 5 & \beta \\ 1 & \gamma & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{or} \quad (A-5I)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} \alpha+2-1=5 \\ 2+10+\beta=10 \\ 1+2\gamma+2=5 \end{matrix}\right] \Rightarrow \left.\begin{matrix} \alpha=4 \\ \beta=-2 \\ \gamma=1 \end{matrix}\right]$$

(03 Marks)
one mark for each.

(b)

$$(A-\lambda I)=0 \Rightarrow \lambda^3-11\lambda^2+39\lambda-45=0$$

or $\lambda_1+\lambda_2=6$, $\lambda_1\lambda_2=9$ — 01 Mark

$$\Rightarrow \lambda_1=\lambda_2=3$$

01 Mark

$$(A-3I)X=0 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1+x_2-x_3=0 \quad \text{---} \ (*)$$

01 Mark

Two L.I. eigenvectors $(1,0,1)^T$ and $(0,1,0)^T$
or two L.I. evectors satisfying $(*)$ — 02 Marks

(c) Yes, A is diagonalizable since A.M. = G.M. for $\lambda=5$ and $\lambda=3$

$$P=\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad P=\begin{bmatrix} 1 & X_1 & X_2 \\ 2 & \downarrow & \downarrow \\ 1 & & \end{bmatrix}$$

where X_1, X_2 are two L.I. evectors satisfying $(*)$

02 Marks

Remark: If P is wrong, then no mark is awarded in (c) part.

Question 2 (a):

$$|f(x,y)-0| = \left|\frac{x^3+3xy^2}{3x^2+y^2}-0\right| = |x|\left(\frac{x^2+3y^2}{3x^2+y^2}\right) \le \sqrt{x^2+y^2}\,\frac{\left((3x^2+y^2)+3(3x^2+y^2)\right)}{(3x^2+y^2)}$$

$$\le 4\sqrt{x^2+y^2} \qquad [\text{Marks } 2]$$

or $$\left|\frac{x^3+3xy^2}{3x^2+y^2}-0\right| \le \frac{|x|}{3}\left(1+\frac{8y^2}{3x^2+y^2}\right) \le \frac{|x|}{3}(1+8) = 3|x| \le 3\sqrt{x^2+y^2}$$

[Marks 2]

or In polar coordinates: $x = r\cos\theta, y = r\sin\theta$.

$$\left|\frac{(r\cos\theta)^3+3r^3\sin^2\theta\cos\theta}{3r^2\cos^2\theta+r^2\sin^2\theta}-0\right| \le |r\cos\theta|\left|\frac{\cos^2\theta+3\sin^2\theta}{3\cos^2\theta+\sin^2\theta}\right|$$

$$\le 3|r\cos\theta| \le 3r.$$

[Marks 2]

For a given $\varepsilon > 0$, choose $\delta = \varepsilon/3$ or $\varepsilon/4$; we get

$$\left|\frac{x^3+3xy^2}{3x^2+y^2}-0\right| < \varepsilon, \text{ whenever } \sqrt{x^2+y^2} < \delta.$$

Hence, the function is continuous at $(0,0)$. [Mark 2].

2 (b). $$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^3/3h^2-0}{h} = \frac{1}{3}$$

[Marks 1]

$$f_x(x,y) = \begin{cases} \dfrac{3x^4-6x^2y^2+3y^4}{(3x^2+y^2)^2}, & \text{if } (x,y)\ne(0,0) \\ \dfrac{1}{3}, & \text{if } (x,y)=(0,0) \end{cases}$$

[Marks 1]

At the point $(0,0)$, choose the path $y = mx$

then $\lim_{(x,y)\to(0,0)} f_x(x,y) = \dfrac{3 - 6m^2 + 3m^4}{(3+m^2)^2}$, which depends on the values on m, hence on the path.

therefore, $f_x(x,y)$ is not continuous at $(0,0)$. [Marks 2]

2(c). $f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k) - 0}{k} = 0$ [Marks $\frac{1}{2}$]

By the definition of differentiability of f at $(0,0)$

$$\lim_{(h,k)\to(0,0)} \frac{\dfrac{h^3+3hk^2}{3h^2+k^2} - \dfrac{1}{3}\cdot h - 0\cdot k}{\sqrt{h^2+k^2}}$$ has to be zero.

[Marks 1]

Choose the path $y = mx$ [ie $k = mh$] [Marks $\frac{1}{2}$]

We are getting:

$$\lim_{h\to 0} \frac{\dfrac{h^3 + 3h(mh)^2}{3h^2 + m^2h^2} - \dfrac{1}{3}h}{\sqrt{h^2+m^2h^2}}$$

$$= \lim_{h\to 0} \frac{\dfrac{1+3m^2}{3+m^2} - \dfrac{1}{3}}{\sqrt{1+m^2}}$$

$$= \cancel{\lim_{h\to 0}} \frac{8m^2}{3\sqrt{1+m^2}\,(3+m^2)}.$$ ~~[Marks~~

Since the limit depends on m; hence on the choice ~~of~~ the paths, therefore f is NOT differentiable at $(0,0)$.

[Marks 2].

3 (a) The square of the distance from a point (x_0, y_0) to the line $2x+y-10=0$ is

$$\left(\frac{2x_0+y_0-10}{\sqrt{2^2+1^2}}\right)^2 = \frac{(2x_0+y_0-10)^2}{5}$$

So we need to minimize

$$f(x,y) = \frac{(2x+y-10)^2}{5} \text{ subject to}$$

$$g(x,y) \equiv \frac{x^2}{4}+\frac{y^2}{9}-1=0$$

①

By Lagrange's multiplier

$$\nabla f = \lambda \nabla g, \qquad g(x,y)=0$$

$$\frac{4}{5}(2x+y-10) = \lambda \frac{2x}{4} = \lambda \frac{x}{2}$$

$$\frac{2}{5}(2x+y-10) = \lambda \frac{2y}{9} = \frac{\lambda y}{9}$$

②

$$\Rightarrow \frac{4\lambda y}{9} = \frac{\lambda x}{2} \Rightarrow \lambda\left(\frac{y}{9}-\frac{x}{8}\right)=0$$

Since $\lambda \neq 0$, $y = 9/8\, x$ — ①

$$\Rightarrow \frac{x^2}{4}+\frac{9^2}{8^2}\frac{x^2}{9}=1 \Rightarrow x = \pm 8/5, \; y = \pm 9/5$$

So Shortest distance $= \dfrac{\left|\frac{16}{5}+\frac{9}{5}-10\right|}{\sqrt{5}} = \sqrt{5}$. $(x,y) = (8/5, 9/5)$ ②

<u>Remarks</u>: You have to minimize $\dfrac{|2x_0+y-10|}{\sqrt{5}}$ subject to $g(x,y)=0$. Therefore it suffices to minimize its square. If you are minimizing directely $2x+y-10$ subject to $g(x,y)=0$, we have deducted the marks.

3(b)

$$F_u = x F_x + y F_y \qquad 1$$

$$F_{uu} = x^2 F_{xx} + y^2 F_{yy} + x F_x + y F_y + 2xy F_{xy} \qquad 1$$

$$F_t = -y F_x + x F_y \qquad 1$$

$$F_{tt} = y^2 F_{xx} + x^2 F_{yy} - y F_y - x F_x - 2xy F_{xy} \qquad 1$$

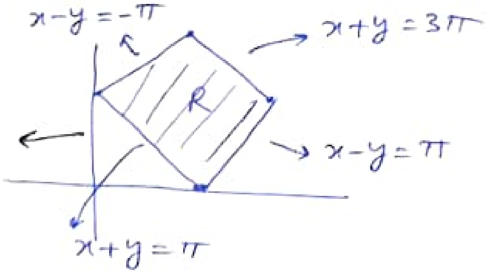
$$\left.\begin{aligned} F_{uu} + F_{tt} &= (x^2+y^2)(F_{xx}+F_{yy}) + 0 \cdot F_{xy} \\ \Rightarrow \phi(u,t) &= e^{2u} \\ \psi(u,t) &= 0 \end{aligned}\right\} \quad 2$$

Remarks: We have deducted marks if some terms like, F_x, F_y, F_{xy} etc or missing in F_{uu} & F_{tt}.

4(a) $\left.\begin{array}{l} u = x+y \\ v = x-y \end{array}\right\} \Rightarrow \begin{array}{l} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{array}$ — 1

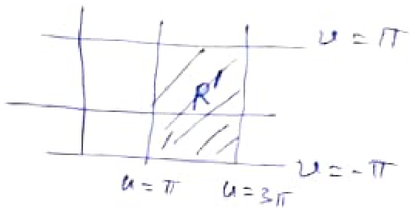
$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$ — 1

$\left.\begin{array}{l} x-y = -\pi \to v = -\pi \\ x-y = \pi \to v = \pi \\ x+y = \pi \to u = \pi \\ x+y = 3\pi \to u = 3\pi \end{array}\right\}$ 2



$$\iint_R (x-y)^{2024} \cos^2(x+y)\, dA$$

$$= \int_{u=\pi}^{3\pi} \int_{v=-\pi}^{\pi} v^{2024} \cos^2 u \left|-\tfrac{1}{2}\right| du\, dv \quad — 1$$



$$= \left(\frac{\pi^{2025}}{2025}\right) \cdot \frac{1}{2} \left(u + \frac{\sin 2u}{2}\right)_{\pi}^{3\pi}$$

$$= \frac{\pi^{2026}}{2025} \quad — 1$$

Note: → '+2' marks is deducted for not computing Jacobian.
→ '2' marks is deducted for not computing limits of u & v properly.
(Finding equations of lines in Rhombus is necessary to find R')
→ final calculations are correct. provided

(b) $I = \int_0^1 y^{q-1} \left(\log\left(\frac{1}{y}\right)\right)^{p-1} dy$

Let $y = e^{-t} \Rightarrow dy = -e^{-t}\, dt$ — 1

$$I = \int_0^\infty e^{-(q-1)t}\, t^{p-1} e^{-t}\, dt$$

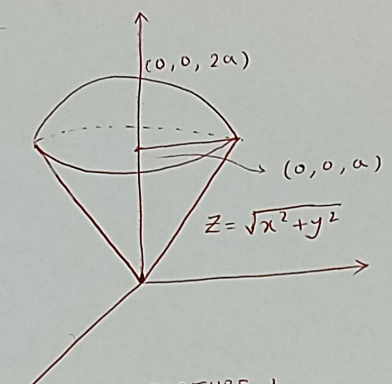
$$= \int_0^\infty e^{-qt}\, t^{p-1}\, dt \quad — 1$$

$\left.\begin{array}{l} qt = z \\ dt = \dfrac{dz}{q} \end{array}\right]$ — 1

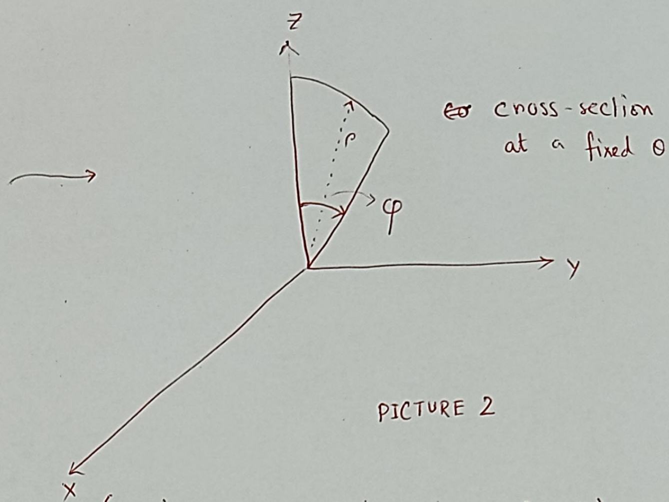
$$= \frac{1}{q^p} \int_0^\infty e^{-z} z^{p-1}\, dz$$

$$= \frac{\Gamma(p)}{q^p} \quad — 1$$

5) a)



PICTURE 1



PICTURE 2

Let the spherical coordinates be $(\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho\cos\varphi)$

For the region : ① $0 \le \theta \le 2\pi$ $\boxed{1/2}$

② $0 \le \varphi \le \pi/4$ $\boxed{1\tfrac{1}{2}}$ ⊛ not shown : Deduct 1

③ $0 \le \rho \le 2a\cos\varphi$ $\boxed{1\tfrac{1}{2}}$ ⊠ not shown : Deduct 1

① Easy to observe.

② The range of φ comes from : $z = \rho\cos\varphi = \rho\sin\varphi = \sqrt{x^2+y^2}$ ⊛
(see Picture 2)
$\Rightarrow \tan\varphi = 1 \Rightarrow \varphi = \pi/4$

③ The range for ρ comes from : $x^2+y^2+(z-a)^2 = a^2$
(see picture 2)
$\Rightarrow \rho^2\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi \cdot a = 0$ ⊠

$\Rightarrow \rho = 2a\cos\varphi$.

So the required volume is given by V and ∴

$$V = \underbrace{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta}_{\boxed{1/2}} = 2\pi \int_0^{\pi/4} \frac{8a^3\cos^3\varphi}{3} \sin\varphi \, d\varphi$$

$$= \frac{16a^3\pi}{3} \cdot \int_0^{\pi/4} \cos^3\varphi \sin\varphi \, d\varphi = \frac{16a^3\pi}{3} \cdot \frac{3}{16}$$

$$= \pi a^3$$ $\boxed{2}$

Remark: Any method other than spherical coordinate;
(Marks given 0)

5) b) We have to determine $I = \iiint\limits_R \rho(x,y,z)\, dx\, dy\, dz$ where R is given by

$$\left(\frac{x}{7}\right)^{2/3} + y^{2/3} + \left(z/4\right)^{2/3} = 1 \quad \text{and} \quad \rho(x,y,z) = 10\,|xyz|.$$

Now $I = 8 I_1$ where $I_1 = \iiint\limits_{R_1} \rho(x,y,z)\, dx\, dy\, dz$. ——— ⊛

$R_1 = R \cap \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}$

We take $u = \left(x/7\right)^{2/3}$, $\quad J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$v = y^{2/3}$

$w = \left(z/4\right)^{2/3}$

$$= 4 \times 7 \times \left(\frac{3}{2}\right)^3 \times \sqrt{uvw} \qquad \boxed{1}$$

R_1 transforms to S_1: $u + v + w = 1$, $u \geq 0$, $v \geq 0$, $w \geq 0$.

So
$$I_1 = \iiint\limits_{R_1} 10\, xyz\, dx\, dy\, dz$$

$$= \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} 10 \cdot 7\, u^{3/2} \cdot v^{3/2} \cdot 4\, w^{3/2} \cdot 4 \times 7 \times \left(\frac{3}{2}\right)^3 u^{1/2}\, v^{1/2}\, w^{1/2}\, dw\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 10 \times 7 \times 4 \times 4 \times 7 \times \left(\frac{3}{2}\right)^3 \times u^2 v^2 w^2\, dw\, dv\, du \qquad \boxed{2}$$

Now we use Dirichlet Integral formula. (Here $a = b = c = 1 = p = q = r$
$\alpha = \beta = \gamma = 3$)

$$= 70 \times \overset{2}{\cancel{16}} \times 7 \times \frac{27}{\cancel{8}} \times \frac{1 \cdot (\Gamma(3))^3}{\Gamma(3+3+3+1)}$$

$$= 70 \times 14 \times 27 \times \frac{8}{\Gamma(10)} = \overset{7}{\cancel{70}} \times 14 \times \overset{3}{\cancel{27}} \times \frac{\cancel{8}}{\underset{\cancel{9!}\ \cancel{8!}\ \cancel{7!}\ 6!}{\cancel{10!}}} \times 10 \qquad \boxed{2}$$

$$= \frac{7 \times 6! \times 10}{6!} = \frac{7}{5!} \times 10$$

$$= \frac{7}{120}$$

$\Rightarrow$ [illegible] $\qquad I = \dfrac{\overset{2}{\cancel{8}} \times 7}{\underset{3}{\cancel{12}}} = \boxed{\dfrac{14}{3}}$

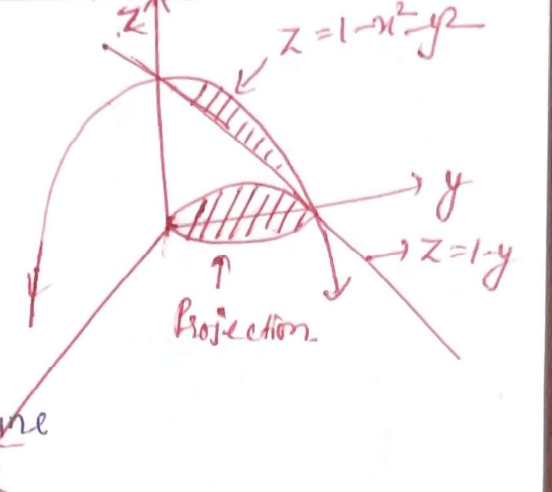
(If this part is missing then deduct only 1/2, provided ⊛ is written somewhere)
If . . . . , ⊛ is not mentioned : Deduct 1.

Q6. (a)

NOTE: 1. If all 3 limits are evaluated correctly then 2 marks
2. If Projection is correct and limit (integral) evaluation is correct 2 marks
3. If final answer is correct without any mistake, 2 marks

To evaluate

$$\iiint_V dz\,dx\,dy$$



Given : $z=1-x^2-y^2$, $z \geq 1-y$

$$V=\iint_R \int_{1-y}^{1-x^2-y^2} dz\,dx\,dy \quad : R \text{ is Projection of region on X-Y plane}$$

$$\Rightarrow V = \iint_R (y-x^2-y^2)\,dx\,dy$$

Now

$1-x^2-y^2=1-y \Rightarrow x^2+y^2=y$

changing into polar coordinate

$x=r\cos\theta,\ y=r\sin\theta$

$\Rightarrow r^2=r\sin\theta \Rightarrow r(r-\sin\theta)\Rightarrow r=0, \sin\theta$

and $-\frac{\pi}{2}\le\theta\le\pi/2$: $dx\,dy = r\,dr\,d\theta$ (by Jacobian) — 2 marks.

$$\therefore V=\int_{-\pi/2}^{\pi/2}\int_0^{\sin\theta}(r\sin\theta-r^2)\,r\,dr\,d\theta$$

$$\Rightarrow V = 2\int_0^{\pi/2}\left[\frac{r^3\sin\theta}{3}-\frac{r^4}{4}\right]_0^{\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{\sin^4\theta}{3}-\frac{\sin^4\theta}{4}\right)d\theta$$

$$= 2\times\frac{1}{12}\int_0^{\pi/2}\sin^4\theta\,d\theta \quad \text{— 2 marks}$$

$$= \frac{1}{12}\left(2\int_0^{\pi/2}\sin^4\theta\,d\theta\right)$$

$$\left(\because 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\beta(m,n)\right)$$

$$= \frac{1}{12}\beta\left(\frac{5}{2},\frac{1}{2}\right)$$

$$\left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\beta(m,n)\right)$$

$$= \frac{1}{12}\,\frac{\Gamma(5/2)\Gamma(1/2)}{\Gamma(3)}$$

$$= 1/12\times\frac{3/2\times 1/2\times\sqrt{\pi}\times\sqrt{\pi}}{2} = \boxed{\frac{\pi}{32}} \quad \text{— 2 marks}$$

Q6 (b) Evaluate the Surface integral

$$\iint_S (x^2+y^2+z)\, ds \quad \text{, over the Surface of the sphere } x^2+y^2+z^2=1$$

Solution

Consider the Spherical coordinate

$$x = \cos\theta\sin\phi,\quad y=\sin\theta\sin\phi,\quad z=\cos\phi$$

$$0 \le \theta \le 2\pi,\quad 0\le \phi \le \pi$$

then

$$f(x,y,z) = x^2+y^2+z = \cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos\phi$$

$$\phi(\theta,\phi) = \sin^2\phi + \cos\phi \quad ; \text{ and } ds = \sin\phi\, d\theta\, d\phi$$

Find ds

2 marks

parametrization

Now

$$I = \iint_S (x^2+y^2+z)\,ds = \int_0^{\pi}\int_0^{2\pi} (\sin^2\phi+\cos\phi)\,\sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi}\left(\int_0^{2\pi} d\theta\right) \sin^3\phi + \cos\phi\sin\phi\, d\phi$$

Put $\cos\phi = t$

then diff. w.r.t.

$-\sin\phi\, d\phi = dt$

$\phi=0 : t=1$

$\phi=\pi : t=-1$

$$= 2\pi \int_0^{\pi} \sin\phi(1-\cos^2\phi)\,d\phi + \int_0^{\pi}\sin\phi\cos\phi\, d\phi$$

$$= 2\pi\left[\int_{-1}^{1}(1-t^2)\,dt + \int_0^{\pi}\sin\phi\, d(\sin\phi)\right]$$

$$= 2\pi\left[\left[t-\frac{t^3}{3}\right]_{-1}^{1} + \left[\frac{\sin^2\phi}{2}\right]_0^{\pi}\right]$$

$$= 2\pi\left[\left\{\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right\}+0\right]$$

$$= 2\pi \times \frac{4}{3}$$

$$= \frac{8\pi}{3}$$

3 marks

Q6)
(b) USING DIVERGENCE THEOREM:

$$\iint_S (x^2+y^2+z)\, ds \quad : S = \{(x,y,z) / x^2+y^2+z^2=1\}$$

$$(x^2+y^2+z) = \vec{F}\cdot\hat{n}$$

Where. $\vec{F} = x\hat{i}+y\hat{j}+\hat{k}$ and $\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$

$$\therefore \quad \vec{F} = x\hat{i}+y\hat{j}+\hat{k}, \quad \hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

1 marks — 1 marks

Using divergence theorem,

$$\oiint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div} F\, dv \quad : V \to \text{Solid region enclosed by the Surface } S.$$

$$= \iiint_V \nabla\cdot(x\hat{i}+y\hat{j}+\hat{k})\, dv$$ (Using Divergence theorem Properly) 2 marks

$$= \iiint_V 2\, dv$$

$$= 2\cdot \iiint_V dv$$

$$= 2 \times \frac{4}{3}\pi (1)^3 \quad \text{(volume of sphere)}$$

$$= \frac{8\pi}{3}$$

(final answer correct) 1 marks

NOTE: 1: If limits and ds is correct 2 marks
2. If integral is evaluated correctly (followed by correct limits) 2 marks
3. If final answer is correct (followed all previous steps correct) 1 mark

IITR

7a $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$\nabla\left(\frac{1}{r}\right)\cdot\hat{r}\Big|_{(1,2,-1)}$ is the directional derivative.

$$\nabla\left(\frac{1}{r}\right) = \sum_{x,y,z} \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\hat{i}$$

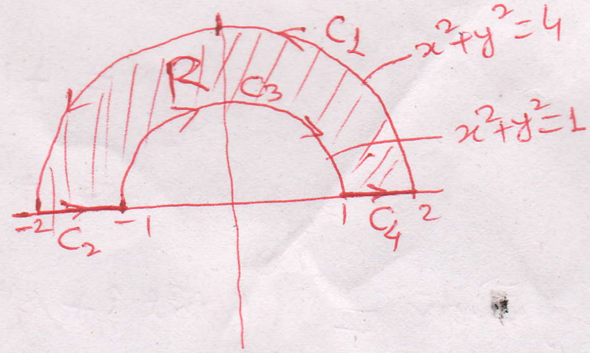
$$= \sum_{x,y,z} -\frac{1}{2(x^2+y^2+z^2)^{3/2}}\cdot 2x\,\hat{i}$$

$$= -\frac{\vec{r}}{r^3}$$ (2) marks

So $\nabla\left(\frac{1}{r}\right)\cdot\hat{r}\Big|_{(1,2,-1)} = -\left(\frac{\vec{r}}{r^3}\right)\cdot\frac{\vec{r}}{r}\Big|_{(1,2,-1)}$

$$= -\frac{1}{r^2}\Big|_{(1,2,-1)} = -\frac{1}{6}$$ (2) marks

7b Parametric forms of



$C_1 : \vec{r}(t) = 2\cos t\, \hat{i} + 2\sin t\, \hat{j}, \quad 0 \le t \le \pi$

$C_3 : \vec{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j}, \quad \pi \le t \le 0$

$C_2 : \; y = 0, \quad -2 \le x \le -1$

$C_4 : \; y = 0, \quad 1 \le x \le 2$

(1) mark

Green's Thm: $\displaystyle\oint_C \vec{F} \cdot dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

where $C = C_1 \cup C_2 \cup C_3 \cup C_4$

$\vec{F} \cdot d\vec{r} = M dx + N dy = y^2 dx + xy\, dy$

$$RHS = \iint_R (y - 2y)\, dx\, dy$$

$$= -\iint_R y\, dx\, dy = -\int_{\theta=0}^{\pi}\int_{r=1}^{2} r\sin\theta \cdot r\, dr\, d\theta$$

$$= -\frac{1}{3}\int_0^{\pi} (r^3)_1^2 \sin\theta\, d\theta$$

$$= +\frac{7}{3}(\cos\theta)_0^{\pi} = -\frac{14}{3}$$ (2) marks

Now

$$\int_{C_1} \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot\frac{dr}{dt}\, dt = \int_{C_1} (y^2\hat{i} + xy\hat{j})\cdot\frac{dr}{dt}\, dt$$

$$= \int_0^{\pi} (4\sin^2 t\,\hat{i} + 4\sin t\cos t\,\hat{j})\cdot(-2\sin t\,\hat{i} + 2\cos t\,\hat{j})\, dt$$

$$= \int_0^{\pi} (-8\sin^3 t + 8\sin t\cos^2 t)\, dt$$

$$= +8\int_0^{\pi} \sin t\,(2\cos^2 t - 1)\, dt$$ Put $\cos t = u$, $-\sin t\, dt = du$

$$= -8\int_1^{-1} (2u^2 - 1)\, du$$

$$= +16\int_0^1 (2u^2 - 1)\, du$$

$$= +16\left[\frac{2}{3}u^3 - u\right]_0^1 = +16\left(\frac{2}{3} - 1\right) = -\frac{16}{3}$$ (2) marks

$$\int_{C_3} \vec{F}\cdot d\bar{r} = \int_{\pi}^{0} (\sin^2 t\,\hat{i} + \sin t\cos t\,\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$$

$$= \int_{\pi}^{0} (-\sin^3 t + \sin t\cos^2 t)\,dt$$

$$= \int_{\pi}^{0} \sin t(2\cos^2 t - 1)\,dt \qquad \text{Put } \cos t = u,\ -\sin t\,dt = du$$

$$= -\int_{-1}^{1} (2u^2 - 1)\,du = -2\int_{0}^{1} (2u^2-1)\,du$$

$$= -2\left[\frac{2}{3}u^3 - u\right]_0^1 = -2\left(\frac{2}{3}-1\right) = \frac{2}{3} \quad (2) \text{ marks}$$

$$\int_{C_2} y^2dx + xy\,dy = \int_{C_4} y^2dx + xy\,dy = 0 \text{ as } y=0 \text{ on } C_2 \text{ and } C_4.$$

8a

Given $\nabla \cdot (f \nabla g) = 2(\nabla f \cdot \nabla g) = 12xyz$

$$f \nabla g = \left(f \frac{\partial g}{\partial x}\right)\hat{i} + \left(f \frac{\partial g}{\partial y}\right)\hat{j} + \left(f \frac{\partial g}{\partial z}\right)\hat{k}$$

So,
$$\nabla \cdot (f \nabla g) = \frac{\partial}{\partial x}\left(f \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f \frac{\partial g}{\partial z}\right)$$
$$= f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$$
$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)$$
$$= f \nabla^2 g + \nabla f \cdot \nabla g$$

2 Marks

(Remark: Direct use of the identity $\nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$ without proof will carry $\boxed{1 \text{ mark}}$ only.)

Then
$$f(\nabla^2 g) = \nabla \cdot (f \nabla g) - \nabla f \cdot \nabla g$$
$$= 12xyz - 6xyz$$
$$f \cdot (\nabla^2 g) = 6xyz$$

$\boxed{1 \text{ mark}}$

Therefore,
$$\nabla(f \nabla^2 g) = \nabla(6xyz)$$
$$= 6(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})$$

$\boxed{1 \text{ mark}}$

Hence,
$$\nabla(f \nabla^2 g)\Big|_{(1,1,1)} = 6(\hat{i} + \hat{j} + \hat{k})$$

$\boxed{1 \text{ mark}}$

8b

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + \lambda yz & xy\cos z + y^2 \end{vmatrix} = \vec{0}$$

1 mark

Solving the above equation, we get

$$\lambda = 2$$ 1 mark

$\vec{F}$ is conservative, so there exists a scalar function f such that

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Therefore

$$\left.\begin{aligned} &\text{(i)} \quad \frac{\partial f}{\partial x} = y\sin z - \sin x \\ &\text{(ii)} \quad \frac{\partial f}{\partial y} = x\sin z + 2yz \\ &\text{(iii)} \quad \frac{\partial f}{\partial z} = xy\cos z + y^2 \end{aligned}\right\} \text{---} (*)$$

1 mark

From (i), we get $f(x,y,z) = xy\sin z + \cos x + \varphi(y,z)$ 1 mark

using (ii), we get $x\sin z + 2yz = x\sin z + \frac{\partial \varphi}{\partial y}$ — (iv)

$$\Rightarrow \frac{\partial \varphi}{\partial y} = 2yz$$

$$\Rightarrow \varphi(y,z) = y^2 z + \psi(z)$$ 1 mark

Thus $f(x,y,z) = xy\sin z + \cos x + y^2 z + \psi(z)$

using (iii), we get $xy\cos z + y^2 = xy\cos z + y^2 + \psi'(z)$

$$\Rightarrow \psi'(z) = 0 \Rightarrow \psi(z) = c$$, where c is a consta[nt]

Hence Potential function $\boxed{f(x,y,z) = xy\sin z + \cos x + y^2 z + c}$, where c is a constant.

1 mark

Remark: (1) If one writes the scalar potential function $f(x,y,z)$ by integrating the three equations in (*) simultaneously and comparing the terms directly in the resulting three equations will carry 2 marks.

(2) ½ Mark will be deducted if one doesn't write the constant c in $f(x,y,z)$

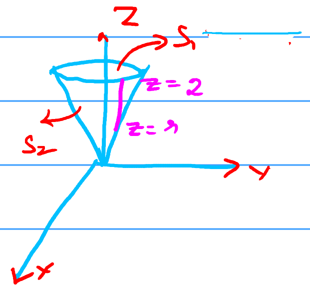
9-a) $\iint_S \bar{F}\cdot\hat{n}\, ds \implies \vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

where $S = \{ \underbrace{z^2 = x^2+y^2}_{S_2} \text{ and } \underbrace{z=2}_{S_1} \}$

By Gauss divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, dV,$$

where $\nabla\cdot\bar{F} = 4z + xz^2 + 3$ (1 mark)



Now

$$I = \int_x \int_y \int_{z=r}^{z=2} (4z + xz^2 + 3)\, dz\, dy\, dx$$ (1 mark)

$$= \int_x \int_y \left(2z^2 + \frac{1}{3}xz^3 + 3z\right)_r^2 dx\, dy$$

$$= \int_x \int_y \left(8 + \frac{8}{3}x + 6 - 2r^2 - \frac{1}{3}xr^3 - 3r\right) dx\, dy$$ (2 marks)

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(14 + \frac{8}{3} r\cos\theta - 2r^2 - \frac{r^3}{3}\cos\theta - 3r\right) r\, dr\, d\theta$$

(using $x = r\cos\theta$, $y = r\sin\theta$, $0<r<2$, $0<\theta<2\pi$)

$$= \int_{r=0}^{r=2} \int_{\theta=0}^{\theta=2\pi} \left(14r + \frac{8}{3} r^2\cos\theta - 2r^3 - \frac{r^4}{3}\cos\theta - 3r^2\right) d\theta\, dr$$

$$= 2\pi \int_{r=0}^{r=2} (14r + 0 - 2r^3 - 0 - 3r^2)\, dr$$

$$= 2\pi \left[7r^2 - \frac{r^4}{2} - r^3\right]_0^2 = 2\pi(28-8-8) = 24\pi \text{ units}$$ (2 marks)

NOTE: 1. After formulation of integral I, if the calculation is wrong (3 marks)

2. If the integral is obtained for S_1, i.e $I = \iint_{S_1} \bar{F}\cdot\hat{n}\, ds = 24\pi$ units, then (2 marks)

If the other surface S_2 is evaluated separately and the value is obtained zero (2 marks)

3. Evaluation using Cartesian method, wrong answer (2 marks)

Correct answer (6 marks)

9b) Use STOKE's Theorem to evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$ where C is boundary curve enclosing $x^2+y^2+z^2 = 4,\ z \geq 0$

STOKE's Theorem $\oint \bar{F}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{F})\cdot\hat{n}\,ds$ —— (1 mark)

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -\hat{i}-\hat{j}-\hat{k}$$ —— (1 mark)

$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and $\phi := x^2+y^2+z^2-4$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \Rightarrow \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$\text{curl}\,\bar{F}\cdot\hat{n}\,ds = \frac{1}{2}(-x-y-z)$ —— (1 mark)

Also, $\hat{n}\cdot\hat{k}\,ds = dx\,dy$ (OR) $\frac{1}{2}(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}\,ds = dy\,dx$

$$\Rightarrow ds = \frac{2}{z}\,dx\,dy$$

Thus

$$I = \iint_S \frac{1}{2}(-x-y-z)\frac{2}{z}\,dx\,dy = -\iint_S \left(\frac{x}{z}+\frac{y}{z}+1\right)dx\,dy$$

$x = r\cos\theta$
$y = r\sin\theta$
$0 < r < 2,\ 0 < \theta < 2\pi$

$$= -\iint_S \left(\frac{x}{\sqrt{4-x^2-y^2}} + \frac{y}{\sqrt{4-x^2-y^2}} + 1\right)dx\,dy$$

$$= -\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\left(\frac{r\cos\theta}{\sqrt{4-r^2}} + \frac{r\sin\theta}{\sqrt{4-r^2}} + 1\right) r\,dr\,d\theta$$

$$= -2\pi\int_{r=0}^{r=2} r\,dr + 0 + 0 = -2\pi\left.\frac{r^2}{2}\right]_0^2 = -4\pi \text{ units}$$

$\longrightarrow$ (2 marks)

NOTE:

1. Statement of STOKE's (1 mark)

For c in xoy plane, $z=0$, $dz=0$

$\Rightarrow I = \int y\,dx$, where $x = 2\cos\theta$, $y = 2\sin\theta$, $dx = -2\sin\theta\,d\theta$

$$= \int 2\sin\theta(-2\sin\theta)\,d\theta = -4\int_0^{2\pi}\sin^2\theta\,d\theta = -4\pi$$

(4 marks)

2. If procedure is correct and calculation wrong (2 marks)